Characterization of the oblique projector $U(VU)^{\dagger}V$ with application to constrained least squares[☆]

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Abstract

We provide a full characterization of the oblique projector $U(VU)^{\dagger}V$ in the general case where the range of U and the null space of V are not complementary subspaces. We discuss the new result in the context of constrained least squares minimization which finds many applications in engineering and statistics.

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1. Introduction

Let $E \in \mathbb{C}^{m\times m}$ be idempotent, $E^2 = E$. The null space and range of any idempotent matrix are complementary, cf. [1, Theorem 2.8],

$$R(E) + N(E) = \mathbb{C}^m, R(E) \cap N(E) = \{0\},$$

and we say that E is an oblique projector onto $R(E)$ along $N(E)$. For any two complementary subspaces of $\mathbb{C}^m$ we denote the oblique projector onto L along M by $P_{L,M}$. The orthogonal projector onto L is denoted by $P_L := P_{L,L^{\perp}}$, where $L^{\perp}$ is the orthogonal complement of L. Oblique projectors arise in numerous engineering and statistical applications, see [1, Chapter 8], [2] and references therein. Many of their properties follow from the general solution to the matrix equation $XAX = X$ studied in 1960-ies in the context of the various pseudoinverses, cf. [3]. This literature is mature, with excellent monographs such as [1]. In particular it is very well understood how to construct an oblique projector with a prescribed range and null space.

Proposition 1.1. *Let L, M be complementary subspaces of $\mathbb{C}^m$. For any two matrices U, V with $R(U) = L$ and $N(V) = M$ one has*

$$P_{L,M} = U(VU)^{\dagger}V,$$

where the superscript "$\dagger$" denotes the Moore-Penrose inverse. If U and V are in addition orthogonal projectors (i.e. they are Hermitian and idempotent) one obtains an even simpler form due to Greville [4, (3.1) and Theorem 2],

$$P_{L,M} = P_L(P_{M^{\perp}}P_L)^{\dagger}P_{M^{\perp}} = (P_{M^{\perp}}P_L)^{\dagger}. \tag{1}$$

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The converse problem of characterizing the range and null space of a given idempotent matrix has not received the same amount of attention. The motivation for studying idempotents of the form $U(VU)^{\dagger}V$ in the general case where $R(U) + N(V) \subsetneq \mathbb{C}^m$ and/or $R(U) \cap N(V) \neq \{0\}$ comes, among others, from constrained least squares optimization with a range of applications mentioned above. Briefly, the problem

$$\min_{x \in \mathbb{C}^n} \|A_1 x - b_1\|^2 , \text{ subject to } A_2 x = b_2,$$

gives rise to the projector $D_2(A_1 D_2)^{\dagger} A_1$ where D_2 is an arbitrary but fixed matrix with the property $R(D_2) = N(A_2)$. In this situation we typically have neither $R(D_2) + N(A_1) = \mathbb{C}^m$ nor $R(D_2) \cap N(A_1) = \{0\}$. Oblique projectors of the form $U(VU)^{\dagger}V$ with $R(U) + N(V) = \mathbb{C}^m$ and $R(U) \cap N(V) \neq \{0\}$ feature also in signal reconstruction, cf. [5].

Given that $U(VU)^{\dagger}V$ has a wide range of applications it is desirable to understand its geometric nature. One might conjecture that in general

$$U(VU)^{\dagger}V = P_{L,M}, \text{ where} \tag{2}$$

$$L = P_{R(U)} N(V)^{\perp} = R(U) \cap (R(U) \cap N(V))^{\perp}, \tag{3}$$

$$M = N(V) + (N(V) + R(U))^{\perp}, \tag{4}$$

but the behaviour of the projector is somewhat more intricate and cannot be described based on the knowledge of $R(U)$ and $N(V)$ alone. The conjecture (2)-(4) turns out to be true only when both U and V are orthogonal projectors. Surprisingly, the main tool in proving the general result is the Zlobec formula [6] in conjunction with Proposition 1.1.

The result presented here is different from the problem discussed by Rao and Yanai [7] in which projectors onto and along two given subspaces are considered under the assumption that the subspaces are not necessarily spanning the whole space. In such a situation, the projector no longer needs to be idempotent.

The paper is organized as follows. In section 2 we introduce required terminology and notation, we establish the main tools and prove Proposition 1.1. In section 3 we state and prove the main result. In section 4 we discuss application of the main result to constrained least squares minimization and the link to the minimal norm solution of Eldén [8].

2. Preliminaries

We use notation of [1]. A^* denotes the conjugate transpose of matrix A. We write $r(A)$, $R(A)$, $N(A)$ for the rank, range and null space of A, respectively. Consider the following relations

$$AXA = A, \tag{I.1}$$

$$XAX = X, \tag{I.2}$$

$$AX = (AX)^*, \tag{I.3}$$

$$XA = (XA)^*. \tag{I.4}$$

We write $X \in A\{i, j, \ldots, k\}$, if X satisfies conditions (I.i), (I.j), ..., (I.k). $A^{\dagger}$ denotes the Moore-Penrose inverse which is the unique element of $A\{1, 2, 3, 4\}$. The following theorem is our main tool.

Theorem 2.1 ([1, Theorem 2.13]). *Let $A \in \mathbb{C}^{m\times n}, \tilde{U} \in \mathbb{C}^{n\times s}, \tilde{V} \in \mathbb{C}^{t\times m}$ and*

$$Z = \tilde{U}(\tilde{V}A\tilde{U})^{(1)}\tilde{V},$$

where $(\tilde{V}A\tilde{U})^{(1)}$ is a fixed but arbitrary element of $(\tilde{V}A\tilde{U})\{1\}$. Then

a) $Z \in A\{1\}$ if and only if $r(\tilde{V}A\tilde{U}) = r(A)$;

b) $Z \in A\{2\}$ and $R(Z) = R(\tilde{U})$ if and only if $r(\tilde{V}A\tilde{U}) = r(\tilde{U})$;

c) $Z \in A\{2\}$ and $N(Z) = N(\tilde{V})$ if and only if $r(\tilde{V}A\tilde{U}) = r(\tilde{V})$;

d) $Z = A^{(1,2)}_{R(\tilde{U}),N(\tilde{V})}$ if and only if $r(\tilde{U}) = r(\tilde{V}) = r(\tilde{V}A\tilde{U}) = r(A)$, where $A^{(1,2)}_{R(\tilde{U}),N(\tilde{V})}$ is the unique element of $A\{1,2\}$ with range $R(\tilde{U})$ and null space $N(\tilde{V})$, also known as the oblique pseudoinverse (cf. [9]).

Corollary 2.2. *The Zlobec formula [6],*

$$A^{\dagger} = A^*(A^*AA^*)^{(1)}A^*, \tag{5}$$

is now obtained by setting $\tilde{U} = \tilde{V} = A^*$ in part d) and arguing $A^{(1,2)}_{R(A^*),N(A^*)} = A^{\dagger}$.

The following is a pre-cursor to the main result in this note. The "if" part appears, for example, in [10, (3.51)].

Corollary 2.3. *$\tilde{U}(\tilde{V}\tilde{U})^{(1)}\tilde{V} = P_{R(\tilde{U}),N(\tilde{V})}$ if and only if $r(\tilde{V}\tilde{U}) = r(\tilde{V}) = r(\tilde{U})$.*

Next we show that the form $U(VU)^{\dagger}V$ covers all idempotent matrices.

Lemma 2.4. *Let $U \in \mathbb{C}^{m\times p}, V \in \mathbb{C}^{q\times m}$. $R(U)$ and $N(V)$ are complementary subspaces of $\mathbb{C}^m$ if and only if $r(U) = r(V) = r(VU)$.*

Proof. If: By Corollary 2.3 $U(VU)^{\dagger}V = P_{R(U),N(V)}$ which implies that $R(U), N(V)$ are complementary.

Only if: i) complementarity implies $\dim(R(U)) + \dim(N(V)) = m$. On rearranging we obtain $r(U) = m - \dim(N(V))$ and by the rank-nullity theorem $r(U) = r(V)$.

ii) Complementarity also implies $R(U) \cap N(V) = \{0\}$ which yields $N(VU) = N(U)$. By rank-nullity theorem we obtain $r(VU) = r(U)$. □

Proposition 2.5. *Matrix $E \in \mathbb{C}^{m\times m}$ is idempotent if and only if there are matrices $U \in \mathbb{C}^{m\times p}, V \in \mathbb{C}^{q\times m}$ such that*

$$E := U(VU)^{\dagger}V. \tag{6}$$

Proof. The 'if' statement follows easily from (6) and (I.2),

$$E^2 = U(VU)^{\dagger}VU(VU)^{\dagger}V = E.$$

The 'only if' part: construct U so that its columns form a basis of $R(E)$ and construct V^* so that its columns form the basis of $N(E)^{\perp}$. This implies $R(U) = R(E), N(V) = N(E)$. Since E is idempotent $R(U), N(V)$ are by construction complementary and from Lemma 2.4 we obtain $r(U) = r(V) = r(VU)$. By Corollary 2.3 $U(VU)^{\dagger}V = P_{R(E),N(E)} = E$. □

Remark 2.6. *A comprehensive characterization of projectors appears in [3]. Proposition 2.5 resembles a result of Mitra [11, Theorem 3a] who shows that all idempotent matrices are of the form $\tilde{U}(\tilde{V}\tilde{U})^{(1,2)}\tilde{V}$ where $(\tilde{V}\tilde{U})^{(1,2)}$ is an arbitrary element of $\tilde{V}\tilde{U}\{1,2\}$. This result is generalized further in [1, Theorem 2.13] to the form $\tilde{U}(\tilde{V}\tilde{U})^{(1)}\tilde{V}$, see Corollary 2.3. Proposition 2.5 goes in the opposite direction in order to avoid the ambiguity associated with $\{1,2\}$-inverses.*

To conclude we provide a proof of Proposition 1.1.

Proof (Proposition 1.1). The first statement follows from the 'only if' part in the proof of Proposition 2.5. The second part follows from identities $(P_{M^\perp}P_L)^\dagger = P_L(P_{M^\perp}P_L)^\dagger = (P_{M^\perp}P_L)^\dagger P_{M^\perp}$, see [1, Exercise 2.57]. □

3. Result

Theorem 3.1. *Given two arbitrary matrices $U \in \mathbb{C}^{m\times p}, V \in \mathbb{C}^{q\times m}$ the matrix $E = U(VU)^\dagger V$ is idempotent with range and null space given by*

$$R(E) = R(UU^*V^*) = R(UU^*V^*V) = R(U) \cap ((UU^*)^\dagger(R(U) \cap N(V)))^\perp, \tag{7}$$

$$N(E) = N(U^*V^*V) = N(UU^*V^*V) = N(V) \oplus (V^*V)^\dagger(R(U) + N(V))^\perp. \tag{8}$$

Proof. By Zlobec's formula (5) with $A = VU$ we obtain

$$E = UU^*V^*(U^*V^*VUU^*V^*)^{(1)}U^*V^*V.$$

Setting $\tilde{U} = UU^*V^*, \tilde{V} = U^*V^*V$ we claim $r(\tilde{U}) = r(\tilde{V}) = r(\tilde{V}\tilde{U}) = r(VU)$. Indeed,

$$r(VU) = r(VUU^*V^*) = r(VUU^*V^*VUU^*V^*) \leq r(U^*V^*VUU^*V^*) = r(\tilde{V}\tilde{U}), \tag{9}$$

$$r(\tilde{V}\tilde{U}) \leq r(\tilde{U}) = r(UU^*V^*) \leq r(U^*V^*) = r(VU), \tag{10}$$

$$r(\tilde{V}\tilde{U}) \leq r(\tilde{V}) = r(U^*V^*V) \leq r(U^*V^*) = r(VU). \tag{11}$$

Corollary 2.3 yields $R(E) = R(\tilde{U}), N(E) = N(\tilde{V})$. From

$$r(VU) = r(VUU^*V^*) = r(VUU^*V^*VUU^*V^*) \leq r(UU^*V^*V) \leq r(U^*V^*) = r(VU),$$

and from (9)-(11) we obtain $r(VU) = r(UU^*V^*) = r(UU^*V^*V)$ which implies $R(UU^*V^*) = R(UU^*V^*V)$. The proof of $N(U^*V^*V) = N(UU^*V^*V)$ proceeds similarly by showing $r(U^*V^*V) = r(UU^*V^*V)$.

To show the last equality in (8) we observe $\mathbb{C}^m = N(V) \oplus R(V^*)$. Since $N(V) \subseteq N(U^*V^*V)$ we have

$$N(U^*V^*V) = N(V) \oplus (R(V^*) \cap N(U^*V^*V)). \tag{12}$$

Continuing with the second term on the right hand side we obtain

$$\begin{aligned} y \in R(V^*) \cap N(U^*V^*V) &\iff (V^*Vy \in N(U^*) \cap R(V^*)) \wedge (y \in R(V^*)) \\ &\iff y \in (V^*V)^\dagger(N(U^*) \cap R(V^*)), \end{aligned}$$

which yields

$$R(V^*) \cap N(U^*V^*V) = (V^*V)^\dagger(R(U)^\perp \cap N(V)^\perp) = (V^*V)^\dagger(R(U) + N(V))^\perp. \tag{13}$$

On substituting (13) into (12) we obtain the desired result.

The last equality in (7) is obtained by writing $R(UU^*V^*) = N(VUU^*)^\perp$ and then evaluating $N(VUU^*)$ by exchanging the role of U and V^* in (12) and (13). □

Remark 3.2. *Special cases of Theorem 3.1 include situations covered by Corollary 2.3 in which $r(U) = r(V) = r(VU)$ and we have $R(E) = R(U), N(E) = N(V)$; the Langenhop form [12, Lemma 2.2] with $VU = I$ is a case in point. The Greville formula (1) also falls into this category. Hirabayashi and Unser [5, Lemma 3] encounter the case $R(U)+N(V) = \mathbb{C}^m$ and $R(U)\cap N(V) \neq \{0\}$, yielding $R(E) = R(UU^*V^*), N(E) = N(V)$.*

4. Application

Proposition 4.1. *Let $A_1 \in \mathbb{C}^{m\times n}, b_1 \in \mathbb{C}^m, A_2 \in \mathbb{C}^{k\times n}, r(A_2) = k \geq 1, b_2 \in \mathbb{C}^k$. Solutions of the problem*

$$\min_{x\in\mathbb{C}^n} \|A_1 x - b_1\|^2 \ , \ \textit{subject to } A_2 x = b_2, \tag{14}$$

lie in the set

$$\Xi = \{D_2(A_1D_2)^\dagger A_1 A_1^\dagger b_1 + (I - D_2(A_1D_2)^\dagger A_1)(A_2^\dagger b_2 + z) : z \in N(A_2)\}, \tag{15}$$

where D_2 is an arbitrary but fixed matrix with the property $R(D_2) = N(A_2)$.

Proof. See [1, Exercise 3.10]. □

In general, the projector $D_2(A_1D_2)^\dagger A_1$ will depend on how D_2 is chosen. However, Theorem 3.1 shows that there is a special case when $D_2(A_1D_2)^\dagger A_1$ is actually invariant to the choice of D_2.

Corollary 4.2. *Using the notation of Proposition 4.1 assume further $r(A_1) = n$. Then*

$$D_2(A_1D_2)^\dagger A_1 = P_{N(A_2),(A_1^*A_1)^{-1}R(A_2^*)},$$

and Ξ is a singleton,

$$\Xi = \{A_1^\dagger b_1 + (A_1^*A_1)^{-1}A_2^*(A_2(A_1^*A_1)^{-1}A_2^*)^{-1}(b_2 - A_2A_1^\dagger b_1)\}.$$

Proof. We have $N(A_1) = 0$ and by Theorem 3.1

$$\begin{aligned} R(D_2(A_1D_2)^\dagger A_1) &= R(D_2) \cap (\{0\})^\perp = N(A_2), \\ N(D_2(A_1D_2)^\dagger A_1) &= (A_1^*A_1)^{-1}R(D_2)^\perp = (A_1^*A_1)^{-1}R(A_2^*). \end{aligned}$$

This implies $(I - D_2(A_1D_2)^\dagger A_1)z = 0$ for all $z \in N(A_2)$ and by Proposition 1.1

$$(I - D_2(A_1D_2)A_1) = (A_1^*A_1)^{-1}A_2^*(A_2(A_1^*A_1)^{-1}A_2^*)^{-1}A_2.$$

The rest follows from Proposition 4.1. □

Note that Corollary 4.2 is not covered by Corollary 2.3 since $n - k = r(D_2) = r(A_1D_2) < r(A_1) = n$. In situations where the choice of D_2 impacts on the projector $D_2(A_1D_2)^\dagger A_1$ Theorem 3.1 guides us to the convenient choice of D_2 which simplifies the geometry of the result and also helps to identify the element of Ξ with minimal distance from a given reference point.

Corollary 4.3. *Using the notation of Proposition 4.1 the following statements hold:*

1. *The constrained least squares minimizer in (14) lies in the set*

$$\Xi = \{A_1^\dagger b_1 + P_{\mathcal{Y},\mathcal{X}}(A_2^\dagger b_2 - A_1^\dagger b_1) + z : z \in N(A_1) \cap N(A_2)\}, \tag{16}$$

with

$$P_{\mathcal{X},\mathcal{Y}} = I - P_{\mathcal{Y},\mathcal{X}} = (A_1(I - A_2^\dagger A_2))^\dagger A_1, \tag{17}$$

$$\mathcal{X} = P_{N(A_2)}R(A_1^*) = N(A_2) \cap (N(A_2) \cap N(A_1))^\perp, \tag{18}$$

$$\mathcal{Y} = N(A_1) \oplus (A_1^*A_1)^\dagger (N(A_1) + N(A_2))^\perp. \tag{19}$$

2. *The element of* Ξ *with the smallest Euclidean norm is given by*

$$\xi := A_1^\dagger b_1 + P_{\mathcal{Y},\mathcal{X}}(A_2^\dagger b_2 - A_1^\dagger b_1).$$

3. *For any* $y \in \mathbb{C}^n$ *the solution of* $\min_{x\in\Xi} \|x - y\|$ *is given by*

$$\psi(y) := \xi + P_{N(A_1)\cap N(A_2)} y. \tag{20}$$

Proof. 1. On setting $D_2 = I - A_2^\dagger A_2 = P_{N(A_2)}$ Proposition 4.1 and Theorem 3.1 yield

$$\Xi = A_1^\dagger b_1 + P_{\mathcal{Y},\mathcal{X}}(A_2^\dagger b_2 - A_1^\dagger b_1 + N(A_2)), \tag{21}$$

with $P_{\mathcal{X},\mathcal{Y}}$, $\mathcal{X}$ and $\mathcal{Y}$ given in (17)-(19). From (18) we obtain $N(A_2) = \mathcal{X} \oplus (N(A_1) \cap N(A_2))$ which implies

$$P_{\mathcal{Y},\mathcal{X}} N(A_2) = P_{\mathcal{Y},\mathcal{X}}(N(A_1) \cap N(A_2)) = N(A_1) \cap N(A_2), \tag{22}$$

the last equality following from $N(A_1) \cap N(A_2) \subseteq \mathcal{Y}$. Substitution of (22) into (21) yields (16).

2. By (18) we have $\mathcal{X} \subseteq (N(A_2) \cap N(A_1))^\perp = R(A_1^*) + R(A_2^*)$. Consequently

$$P_{\mathcal{Y},\mathcal{X}}(R(A_1^*) + R(A_2^*)) = (I - P_{\mathcal{X},\mathcal{Y}})(R(A_1^*) + R(A_2^*)) \subseteq R(A_1^*) + R(A_2^*). \tag{23}$$

This implies

$$\xi \in R(A_1^*) + R(A_2^*) = (N(A_2) \cap N(A_1))^\perp. \tag{24}$$

By (16) $x - \xi \in N(A_1) \cap N(A_2)$ for any $x \in \Xi$ which together with (24) yields

$$\|x\|^2 = \|x - \xi + \xi\|^2 = \|x - \xi\|^2 + \|\xi\|^2 \text{ for all } x \in \Xi.$$

3. By (16), (20) and (24) we obtain $x - \psi(y) \in N(A_2) \cap N(A_1)$ and $\psi(y) - y \in (N(A_2) \cap N(A_1))^\perp$ which implies $\|x - y\|^2 = \|x - \psi(y) + \psi(y) - y\|^2 = \|x - \xi\|^2 + \|\xi - y\|^2$, for all $x \in \Xi$. □

Remark 4.4. *It is well known that vector* $A_1^\dagger b_1$ *has the smallest Euclidean norm among all solutions of the unconstrained least squares problem* $\min_{x\in\mathbb{C}^n} \|A_1 x - b_1\|$. *We have shown in part 2. of Corollary 4.3 that* $\xi = A_1^\dagger b_1 + P_{\mathcal{Y},\mathcal{X}}(A_2^\dagger b_2 - A_1^\dagger b_1)$ *is the shortest solution of the constrained least squares problem (14).*

Eldén [8, Theorem 2.1] studied minimal norm solutions of constrained least squares. On setting

$$h = b_2 - A_2 A_1^\dagger b_1, \quad f = x - A_1^\dagger b_1, \quad K = A_1, \quad L = A_2, \quad M = I,$$

Eldén's solution yields that

$$\zeta := A_1^\dagger b_1 + (I - P_{N(A_2)}(A_1 P_{N(A_2)})^\dagger A_1) A_2^\dagger (b_2 - A_2 A_1^\dagger b_1)$$

minimizes the Euclidean distance $\|x - A_1^\dagger b_1\|$ *among all constrained minimizers* $x \in \Xi$.

With a little bit of work one finds $\zeta = \xi - P_{\mathcal{Y},\mathcal{X}} P_{N(A_2)} A_1^\dagger b_1 = \xi$, *since* $P_{N(A_2)} A_1^\dagger \in \mathcal{X}$ *by virtue of (18). Thus part 3. of Corollary 4.3 simplifies and extends Eldén's result.*

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